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Truncated Exponential Log-Topp-Leon Generalized Family of Distributions: Properties and Application to Real Data sets

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ABSTRACT

Modeling real-life data is still a major problem, especially in situations where the data does not follow a certain distribution. This issue still leaves a gap for the need to propose a new model, which is expected to address the attributes and behavior of real data. Based on this, a novel generalized family of distributions called "truncated exponential log Topp-Leone Generalized Families of Distributions" is introduced based on Topp-Leone distribution and truncated exponential distribution. Along with its features, the new generator's statistical qualities are examined. Also, the generator parameters and parameter vector of the baseline distribution were calculated with the maximum likelihood, goodness of fit (Cramer-Von-Mises), and least square techniques. We demonstrate how adaptable the suggested approach is by applying it to two data sets (waiting time data sets and Wheaton River flood data sets) and conclude that the TELTL-G outperforms other comparable distributions in terms of fit. In conclusion, we advise that this generator be used as a generalized family of distributions in modeling time failure.

1. INTRODUCTION

Modelling Statistical data has become an important aspect of lifetime data [10] in many areas. More statistical models that offer trustworthy and accurate predictions of the underlying processes are being developed by researchers in various fields as data sets get more diverse and complicated [20]. This attracts many researchers in many field including reliability analysis, engineering, economics, biological studies, and environmental and medical sciences, which require a suitable statistical model for accurate data realization.

According to [25], there is a pressing need to suggest new models that can more accurately represent the real-life phenomena present in a given dataset because there are still numerous situations in which none of the conventional or classical probability models can adequately describe the actual data. According to [22], the concept essentially began with the definition of various mathematical functional forms, followed by the introduction of location, scale, or inequality parameters. In order to make the distribution of interest more flexible, distribution theory researchers typically model data by creating a new distribution family or by adding a new parameter [4]. It has been demonstrated that this induction of parameter (s) is helpful for investigating tail properties and for enhancing the suggested generator family's goodness-of-fit [18].

[24] Created a novel J-shaped histogram based on experimental distribution. It is a bounded support continuous distribution that can be used to simulate a distribution's lifetime. There has been little discussion about the Topp Leone (TL) distribution prior to its discovery, it was unknown until [14] examined a few of its characteristics, moments, and central moments. The flexibility of its hazard rate function (HRF) makes TL a suitable distribution for modeling lifetime data, which can be either bathtub or non-increasing, depending on the parameter's values [13].

Many researchers have introduced a generalization of distribution , like: Topp-Leone Generalized Family of Distribution TLG by [11], A New Topp-Leone Generalized Family of Distribution by [9], Topp-Leone Exponentiated-Generalized by [19], Sin Topp-Leone Generalized Distribution by [2], Exponentiated Topp-Leone Exponentiated-Generalized Distibution by [17], Frechet Topp-Leone Generalized Distribution by [17], Transmuted Topp-Leone Generalized by [26], New Power Topp-Leone Generated Distribution by [6], and Poisson Topp-Leone Generator of Distribution by [12] among others.

In this paper, we introduce a new generalization of distribution called truncated exponential log topp-leone generalized family of distributions, which comprises the pdf of a truncated exponential distribution and a log-topp-leone generalized family of distributions. The cdf and pdf of the log topp-leone generalized distribution are respectively given by;

$$F_{LTL-G}(y) = \left(1 - e^{-2H(y,\psi)}\right)^{\theta} \quad y, \theta > 0 \qquad (1)$$

And the pdf is

$$f_{LTL-G}(y) = 2\theta e^{-2H(y,\psi)}\left(1 - e^{-2H(y,\psi)}\right)^{\theta-1} \quad y, \theta > 0 \qquad (2)$$

The proposed Truncated exponential Log Topp-leone Generalized Family of distributions is drive from the cdf of truncated exponential distribution and the cdf of Log Topp-Leone G distribution in equation (1) using the link function by integrating the cdf of truncated exponential distribution with limit from 0 to the cdf of log topp-leone generalized family, and is drive as follows.

$$F_{TELTL-G}(y,\beta,\theta,\psi) = \int_0^{\left(1-e^{-2H(y,\psi)}\right)^{\theta}} \frac{\beta e^{-\beta y}}{1-e^{-\beta}}\, dy \qquad (3)$$

$$F_{TELTL-G}(y,\beta,\theta,\psi) = \frac{1-e^{-\beta\left(1-e^{-2H(y,\psi)}\right)^{\theta}}}{1-e^{-\beta}} \qquad (4)$$

And we can find the probability density function by differentiating the CDF using quotient rule.

$$f_{TELTL-G}(y,\beta,\theta,\psi) = \frac{2\beta\theta\left(1-e^{-2H(y,\psi)}\right)^{\theta-1}}{1-e^{-\beta}} \text{x}$$

$$h(y,\psi)e^{-2H(y,\psi)}e^{-\beta\left(1-e^{-2H(y,\psi)}\right)^{\theta}} \quad y, \theta, \beta > 0 \qquad (5)$$

here $h(y)$ and $H(y)$ are the probability density function and cumulative density function of the baseline distribution, β is a shape parameter, θ is a second shape parameter, and ψ is the parameter vector of the baseline distribution.

1.1 Important expansion of TELTL-G

Given the probability density function:

$$f_{TELTL-G}(y,\beta,\theta,\psi) = \frac{2\beta\theta\left(1-e^{-2H(y,\psi)}\right)^{\theta-1}}{1-e^{-\beta}} x$$

$$h(y,\psi)e^{-2H(y,\psi)}e^{-\beta\left(1-e^{-2H(y,\psi)}\right)^{\theta}} \quad y,\theta,\beta > 0 \quad (6)$$

Note that $\left(1-e^{-2H(y,\psi)}\right)^{\theta-1} < 1$ and β≠ 1

Using power series expansion

$$\left(1-e^{-2H(y,\psi)}\right)^{\theta-1} =$$

$$\sum_{i=0}^{\infty}\binom{\theta-1}{i}(-1)^i e^{-2iH(y,\psi)} \quad (7)$$

Substituting (7) in to (6), we have

$$f(y,\beta,\theta,\psi) = \sum_{i=0}^{\infty}\binom{\theta-1}{i}\frac{2\beta\theta(-1)^i h(y,\psi)}{1-e^{-\beta}}\text{x}$$

$$e^{-2(i+1)H(y,\psi)}e^{-\beta\left(1-e^{-2H(y,\psi)}\right)^{\theta}} \quad (8)$$

Also, using maclaurin series expansion

$$e^{-2(i+1)H(y,\psi)} = \sum_{j=0}^{\infty}\frac{(-1)^j 2^j (i+1)^j \left(H(y,\psi)\right)^j}{j!} \quad (9)$$

Substitute (9) in to (8), we have

$$f(y,\beta,\theta,\psi) = \sum_{i-1}^{\infty}\sum_{j=0}^{\infty}\binom{\theta-1}{i}\frac{2^{j+1}(-1)^{i+j}(i+1)^j}{(1-e^{-\beta})j!}\text{x}$$

$$\beta\theta h(y,\psi)\left(H(y,\psi)\right)^j e^{-\beta\left(1-e^{-2H(y,\psi)}\right)^{\theta}} \quad (10)$$

Likewise,

$$e^{-\beta\left(1-e^{-2H(y,\psi)}\right)^{\theta}} = \sum_{k=0}^{\infty}\frac{(-1)^k 2\beta^k\left(1-e^{-2H(y,\psi)}\right)^{k\theta}}{k!} \quad (11)$$

Putting (11) in to (10), equation (10) become

$$f(y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{\theta-1}{i}\frac{2^{j+1}(-1)^{i+j+k}(i+1)^j}{(1-e^{-\beta})j!k!}\text{x}$$

$$\beta^{k+!}\theta h(y,\psi)\left(H(y,\psi)\right)^j\left(1-e^{-2H(y,\psi)}\right)^{k\theta} \quad (12)$$

Again,

$$\left(1-e^{-2H(y,\psi)}\right)^{k\theta} = \sum_{m=0}^{\infty}\binom{k\theta}{m}\frac{e^{-2mH(y,\psi)}}{m!} \quad (13)$$

Implies that, equation (12) become.

$$f(y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\binom{\theta-1}{i}\binom{k\theta}{m}\text{x}$$

$$\theta h(y,\psi)\left(H(y,\psi)\right)^j e^{-2mH(y,\psi)}\text{x}$$

$$\frac{2^{j+1}(i+1)^j(-1)^{i+j+k}\beta^{k+1}}{(1-e^{-\beta})j!k!} \quad (14)$$

Similarly,

$$e^{-2H(y,\psi)} = \sum_{p=0}^{\infty}\frac{(-1)^p 2^p m^p H(y,\psi)^p}{p!} \quad (15)$$

By substituting (15) in to (14), we have.

$$f(y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\binom{\theta-1}{i}\binom{k\theta}{m}\text{x}$$

$$\frac{2^{j+p+1}(-1)^{i+j+k+p}(i+1)^j\beta^{k+1}\theta m^p h(y,\psi)\left(H(y,\psi)\right)^{j+p}}{(1-e^{-\beta})j!k!} \quad (16)$$

$$\Rightarrow f(y,\beta,\theta,\psi) = \mathcal{T}h(y,\psi)\left(H(y,\psi)\right)^{j+p}$$

Where,

$$\mathcal{T} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\binom{\theta-1}{i}\binom{k\theta}{m}\text{x}$$

$$\frac{2^{j+p+1}(-1)^{i+j+k+p}(i+1)^j\beta^{k+1}\theta m^p}{(1-e^{-\beta})j!k!} \quad (17)$$

2. SOME PROPERTIES OF TELTL-G

In this part, we discussed some mathematical properties of TELTL-G as follows:

2.1. Quantile of TELTL-G

Let Y be the TELTL-G random variable with pdf and cdf in (4) and (5), the quantile function of X, say $H^{-1}(y,\psi)$ is drive as follows:

$$\upsilon = F_{TELTL-G}(y) = \frac{1-e^{-\beta\left(1-e^{-2H(y,\psi)}\right)^{\theta}}}{1-e^{-\beta}} \quad (18)$$

$$\Rightarrow H^{-1}(y,\psi) = -\frac{1}{2}\left\{1-\left\{\frac{-\ln\left(1-\upsilon(1-e^{-\beta})\right)}{\beta}\right\}^{\frac{1}{\beta}}\right\} \quad (19)$$

Hence, the median (quantile) of the Truncated exponential log top-leone generalized family of distribution when $\upsilon = 0.5$ is given by;

$$y_{v} = -\frac{1}{2}\left\{1 - \left\{\frac{-\ln\left(1-(0.5)\left(1-e^{-\beta}\right)\right)}{\beta}\right\}^{\frac{1}{\beta}}\right\} \quad (20)$$

2.2. Survival function

The Survival function $S(Y)$ of a TELTL-G as one of the important tools for measuring the failure time of a system is given by;

$$S(y) = 1 - F(y) = 1 - \frac{1-e^{-\beta\left(1-e^{-2H(y,\psi)}\right)^{\theta}}}{1-e^{-\beta}} \quad (21)$$

$$\Rightarrow S(y) = \frac{e^{-\beta\left(1-e^{-2H(y,\psi)}\right)^{\theta}} - e^{-\beta}}{1-e^{-\beta}} \quad (22)$$

2.3. Hazard function

The hazard rate function H(Y) of a TELTL-G is given by;

$$H(y) = \frac{f(y)}{S(y)} = \frac{f(y)}{1-F(y)} \quad (23)$$

$$\Rightarrow H(y) = \frac{2\beta\theta\left(1-e^{-2H(y,\psi)}\right)^{\theta-1} h(y,\psi)e^{-2H(y,\psi)}}{e^{-\beta\left(1-e^{-2H(y,\psi)}\right)^{\theta}} - e^{-\beta}} \text{x}$$

$$e^{-\beta\left(1-e^{-2H(y,\psi)}\right)^{\theta}} \quad (24)$$

2.4. Entropy

Entropy in many situations used to measures the system's randomness, and variation or uncertainty of a random variable Y, and is define as;

$$I_x(y) = \frac{1}{1-\delta} log \int_{-\infty}^{\infty} h(y)^{\delta} dy \quad (25)$$

For the Truncated exponential log top-leone generalized family of distribution, the entropy is given by;

$$I_x(y) = \frac{1}{1-\delta} log \int_0^{\infty} f(y)^{\delta} dy \quad (26)$$

But $f(y)^{\delta} = (\mathcal{T}\Phi)^{\delta}$

$$\Rightarrow I_x(y) = \frac{1}{1-\delta} log \int_0^{\infty} (\mathcal{T}\Phi)^{\delta} dy \quad (27)$$

$$\Rightarrow \frac{1}{1-\delta}\left[\delta log\mathcal{T} + log \int_0^{\infty} \Phi^{\delta} dy\right] \quad (28)$$

Where,

$$\Phi = h(y,\psi)\left(H(y,\psi)\right)^{j+p}$$

2.5. Moment of TELTL-G

Moments is a crucial part of any statistical study. [9] They may be used to characterize key distributional features and forms, such as dispersion and spread as determined by mean and variance and peakness of the distribution as determined by kurtosis. They can also be used to look at the symmetry of the distribution's shape as determine by skewness. The rth moment of TELTL-G distribution is given by:

$$E(y^r) = \mu^r \int_{-\infty}^{\infty} y^r\ f(y,\beta,\theta,\psi) dy \quad (29)$$

$$\mu^r = \mathcal{T} \int_0^{\infty} y^r\, h(y,\psi)\left(H(y,\psi)\right)^{j+p} dy \quad (30)$$

2.6. Moment generating function of TELTL-G

The moment generating function of the random variable that follows TELTL-G having pdf in equation (5) is given by;

$$E(e^{ty}) = M_y(t) = \int_{-\infty}^{\infty} e^{ty} f(y,\beta,\theta,\psi) dy \quad (31)$$

Using power series expansion.

$$e^{ty} = \sum_{q=0}^{\infty} \frac{t^q y^q}{q!} \quad (32)$$

Substitute equation (32) in to (31), we have.

$$M_y(t) = \sum_{q=0}^{\infty} \int_{-\infty}^{\infty} \frac{\mathcal{T} t^q y^q h(y,\psi)\left(H(y,\psi)\right)^{j+p}}{q!} dy \quad (33)$$

$$M_y(t) = \sum_{q=0}^{\infty} \mathcal{T}\Delta \quad (34)$$

Where,

$$\Delta = \int_{-\infty}^{\infty} y^q h(y,\psi)\left(H(y,\psi)\right)^{j+p} dy$$

$$\mathcal{T} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\binom{\theta-1}{i}\binom{k\theta}{m}\text{x}$$

$$\frac{2^{j+p+1}(-1)^{i+j+k+p}(i+1)^{j}\beta^{k+1}\theta m^{p}t^{q}}{(1-e^{-\beta})j!\,k!\,q!}$$

2.7. Order statistics

Let $y_1, y_2, y_3, \ldots, y_n$ be a random sample from the TELTL-G distribution and let $\boldsymbol{y(1)}, \ldots, \boldsymbol{y(n)}$ be the corresponding order statistics. The pdf of nth order statistic can be written as;

$$\boldsymbol{f_{i,n}(y) = \frac{n!}{(i-1)(n-i)!} f(y)[F(y)]^{i-1}}\text{x}$$

$$\boldsymbol{[1-F(y)]^{n-i}} \qquad (35)$$

Now, using power series expansion;

$$\boldsymbol{[1-F(y)]^{n-i} = \sum_{j=0}^{n-i}(-1)^{j}\binom{n-i}{j}[F(y)]^{j}} \qquad (36)$$

Implies that equation (35) become;

$$\boldsymbol{f_{i,n}(y) = \frac{(-1)^{j}n!f(y)}{(i-1)(n-i)!}\sum_{j=0}^{n-i}\binom{n-i}{j}[F(y)]^{i+j-1}} \qquad (37)$$

$$\boldsymbol{f_{i,n}(y) = \sum_{j=0}^{n-i}\frac{(-1)^{j}n!f(y)}{(i-1)(n-i-j)!j!}[F(y)]^{i+j-1}} \qquad (38)$$

$$\boldsymbol{f_{i,n}(y) = \sum_{j=0}^{n-i}\frac{\beta\theta(-1)^{j}n!e^{-2H(y,\psi)}\left(1-e^{-2H(y,\psi)}\right)^{\theta-1}}{2^{-1}(i-1)(n-i-j)!j!(1-e^{-\beta})^{i+j}}}\text{x}$$

$$\boldsymbol{h(y,\psi)e^{-\beta\left(1-e^{-2H(y,\psi)}\right)^{\theta}}\left(1-e^{-\beta\left(1-e^{-2H(y,\psi)}\right)^{\theta}}\right)^{i+j-1}} \qquad (39)$$

$$\boldsymbol{f_{i,n}(y) = \sum_{j=0}^{n-i}\frac{2\eta\beta\theta(-1)^{j}n!}{(i-1)(n-i-j)!j!(1-e^{-\beta})^{i+j}}} \qquad (40)$$

Where

$$\boldsymbol{\eta = \frac{h(y,\psi)e^{-2H(y,\psi)}\left(1-e^{-2H(y,\psi)}\right)^{\theta-1}}{e^{\beta\left(1-e^{-2H(y,\psi)}\right)^{\theta}}\left(1-e^{-\beta\left(1-e^{-2H(y,\psi)}\right)^{\theta}}\right)^{1-i-j}}}$$

The minimum order statistics is;

$$\boldsymbol{f_{1,n}(y) = \frac{n!}{(n-1)!}f(y)[1-F(y)]^{n-1}} \quad \textbf{(41)}$$

$$\boldsymbol{= n!\,f(y)[1-F(y)]^{n-1}}$$

$$\boldsymbol{f_{1,n}(y) = \sum_{j=0}^{\infty}\binom{n-1}{j}n!\,(-1)^{j}f(y)[F(y)]^{j}} \qquad (42)$$

Substituting f(y) and F(y), we have;

$$\boldsymbol{f_{1,n}(y) = \sum_{j=0}^{\infty}\frac{2\beta\theta(-1)^{j}n!e^{-2H(y,\psi)}\left(1-e^{-2H(y,\psi)}\right)^{\theta-1}}{(1-e^{-\beta})^{j+1}}}\text{x}$$

$$\boldsymbol{\binom{n-1}{j}e^{-\beta\left(1-e^{-2H(y,\psi)}\right)^{\theta}}\left(1-e^{-\beta\left(1-e^{-2H(y,\psi)}\right)^{\theta}}\right)^{j}} \qquad (43)$$

And for the maximum order statistics, equation (35) reduced to;

$$\boldsymbol{f_{i,n}(y) = \frac{n!}{(n-1)!}f(y)[F(y)]^{n-1}} \qquad (44)$$

$$\boldsymbol{f_{n,n}(y) = \sum_{j=0}^{\infty}\frac{2\beta\theta(-1)^{j}n!e^{-2H(y,\psi)}\left(1-e^{-2H(y,\psi)}\right)^{\theta-1}}{(1-e^{-\beta})^{n}}}\text{x}$$

$$\boldsymbol{e^{-\beta\left(1-e^{-2H(y,\psi)}\right)^{\theta}}\left(1-e^{-\beta\left(1-e^{-2H(y,\psi)}\right)^{\theta}}\right)^{n-1}} \qquad (45)$$

3. ESTIMATION

3.1 Maximum Likelihood Estimation

Let $y_1, y_2, y_3, \ldots, y_n$ be a random sample from the TELTL-G family of distribution with pdf in equation (5) with $\boldsymbol{\overline{\omega} = (\beta, \theta, \psi)}$, the TELTL-G's n sample log-likelihood is drive as:

$$\boldsymbol{l = \log(y/\overline{\omega}) = log\prod_{i=1}^{n} f(y/\overline{\omega})} \qquad (46)$$

$$\boldsymbol{l(\overline{\omega}) = \prod_{i=1}^{n}\left[\frac{2\beta\theta\left(1-e^{-2H(y,\psi)}\right)^{\theta-1}h(y,\psi)e^{-2H(y,\psi)}}{1-e^{-\beta}}\right]}\text{x}$$

$$\boldsymbol{e^{-\beta\left(1-e^{-2H(y,\psi)}\right)^{\theta}}} \qquad (47)$$

$$\boldsymbol{logl(\overline{\omega}) = nlog(2) + nlog(\beta) + nlog(\theta) +}$$

$$\boldsymbol{\sum_{i=1}^{n} logh(y,\psi) - 2\sum_{i=1}^{n} H(y,\psi) + nlog(1) +}$$

$$\boldsymbol{(n\theta-1)\log\left(1-e^{-2H(y,\psi)}\right) - \beta\sum_{i=1}^{n}\left(1-e^{-2H(y,\psi)}\right)^{\theta} - nlog(1-e^{-\beta})} \qquad (48)$$

By differentiating the log likelihood with respect to β, θ, and ψ, we have;

$$\frac{dl}{d\beta}=\frac{n}{\beta}-\frac{ne^{-\beta}}{(1-e^{-\beta})}-\sum_{i=1}^{n}\left(1-e^{-2H(y,\psi)}\right)^{\theta} \quad (49)$$

$$\frac{dl}{d\theta}=\frac{n}{\theta}+n\log\left(1-e^{-2H(y,\psi)}\right)-\beta\sum_{i=1}^{n}\left(1-e^{-2H(y,\psi)}\right)^{\theta}\log\left(1-e^{-2H(y,\psi)}\right) \quad (50)$$

$$\frac{dl}{d\psi}=\sum_{i=1}^{n}\frac{h\prime(y,\psi)}{h(y,\psi)}-2\sum_{i=1}^{n}h(y,\psi)+\frac{2(n\theta-1)e^{-2H(y,\psi)}h(y,\psi)}{\left(1-e^{-2H(y,\psi)}\right)}-2\beta\theta h(y,\psi)\sum_{i=1}^{n}\left(1-e^{-2H(y,\psi)}\right)^{\theta-1}e^{-2H(y,\psi)} \quad (51)$$

Where $\frac{dH(y,\psi)}{d\psi}=h(y,\psi)$

3.2. Least Square Estimation

Another method used for estimating parameters of the probability model is least square [21]. Since it is not always possible to acquire the explicit forms of the maximum likelihood estimators, alternative approaches are created to address this issue. Let $y_1,y_2,y_3,\ldots,y_n$ represent the ordered samples from the TELTL-G distribution that were taken from a sample of size n.

$$T(k)=\sum_{i=0}^{n}\left\{F(y)-\frac{i}{n+1}\right\}^{2} \quad (52)$$

$$T(k)=\sum_{i=0}^{n}\left\{\frac{1-e^{-\beta\left(1-e^{-2H(y,\psi)}\right)^{\theta}}}{1-e^{-\beta}}-\frac{i}{n+1}\right\}^{2} \quad (53)$$

The estimate of $\bar{k}_{LSE}=(\bar{\beta},\bar{\theta},\bar{\psi})$can be obtained by differentiating equation (53)

$$\frac{dT(k)}{d\theta}=-2\sum_{i=1}^{n}\left\{\frac{1-e^{-\beta\left(1-e^{-2H(y,\psi)}\right)^{\theta}}}{1-e^{-\beta}}-\frac{i}{n+1}\right\}\left\{\frac{\beta\left(1-e^{-2H(y,\psi)}\right)^{\theta}\log\left(1-e^{-2H(y,\psi)}\right)e^{-\beta\left(1-e^{-2H(y,\psi)}\right)^{\theta}}}{1-e^{-\beta}}\right\} \quad (54)$$

$$\frac{dT(k)}{d\beta}=-2\sum_{i=1}^{n}\left\{\frac{1-e^{-\beta\left(1-e^{-2H(y,\psi)}\right)^{\theta}}}{1-e^{-\beta}}-\frac{i}{n+1}\right\}\mathrm{x}\left\{\frac{(1-e^{-\beta})\left(1-e^{-2H(y,\psi)}\right)^{\theta}e^{-\beta\left(1-e^{-2H(y,\psi)}\right)^{\theta}}}{(1-e^{-\beta})^{2}}-\frac{e^{-\beta}\left(1-e^{-\beta\left(1-e^{-2H(y,\psi)}\right)^{\theta}}\right)}{(1-e^{-\beta})^{2}}\right\} \quad (55)$$

$$\frac{dT(k)}{d\psi}=-2\sum_{i=1}^{n}\left\{\frac{1-e^{-\beta\left(1-e^{-2H(y,\psi)}\right)^{\theta}}}{1-e^{-\beta}}-\frac{i}{n+1}\right\}\mathrm{x}\left\{\frac{2\beta\theta h(y,\psi)e^{-2H(y,\psi)}\left(1-e^{-2H(y,\psi)}\right)^{\theta-1}e^{-\beta\left(1-e^{-2H(y,\psi)}\right)^{\theta}}}{(1-e^{-\beta})}\right\} \quad (56)$$

3.3 Estimation Using Goodness of Fit (Cramer-Von-Mises)

As demonstrated, the minimum biased when compared to the other Cramer-von-Mises goodness-of-fit statistics estimators is Cramer-von-Mises estimates. [7] Provides the formula $C(\kappa)$, where the estimators guarantee its minimum about the unidentified parameters.

$$CV(k)=\frac{1}{12n}+\sum_{i=0}^{n}\left\{F(y)-\frac{2i-1}{2n}\right\}^{2} \quad (57)$$

By solving the above equation, the estimate of k can be obtain;

$$CV(k)=\frac{1}{12n}+\sum_{i=0}^{n}\left\{\frac{1-e^{-\beta\left(1-e^{-2H(y,\psi)}\right)^{\theta}}}{1-e^{-\beta}}-\frac{2i-1}{2n}\right\}^{2} \quad (58)$$

$$\frac{dCV(k)}{d\theta}=-2\sum_{i=1}^{n}\left\{\frac{1-e^{-\beta\left(1-e^{-2H(y,\psi)}\right)^{\theta}}}{1-e^{-\beta}}-\right.$$

$$\frac{2i-1}{2n}\Bigg)\Bigg\{\frac{\beta\left(1-e^{-2H(y,\psi)}\right)^{\theta}\log(1-e^{-2H(y,\psi)})e^{-\beta\left(1-e^{-2H(y,\psi)}\right)^{\theta}}}{1-e^{-\beta}}\Bigg\} \quad (59)$$

$$\frac{dCV(k)}{d\beta} = -2\sum_{i=1}^{n}\left\{\frac{1-e^{-\beta\left(1-e^{-2H(y,\psi)}\right)^{\theta}}}{1-e^{-\beta}} - \frac{2i-1}{2n}\right\}\text{x}$$
$$\left\{\frac{(1-e^{-\beta})\left(1-e^{-2H(y,\psi)}\right)^{\theta}e^{-\beta\left(1-e^{-2H(y,\psi)}\right)^{\theta}}}{(1-e^{-\beta})^2} - \frac{e^{-\beta}(1-e^{-\beta\left(1-e^{-2H(y,\psi)}\right)^{\theta}})}{(1-e^{-\beta})^2}\right\} \quad (60)$$

$$\frac{dCV(k)}{d\psi} = -2\sum_{i=1}^{n}\left\{\frac{1-e^{-\beta\left(1-e^{-2H(y,\psi)}\right)^{\theta}}}{1-e^{-\beta}} - \frac{2i-1}{2n}\right\}\text{x}$$
$$\left\{\frac{2\beta\theta \boldsymbol{h(y,\psi)} e^{-2H(y,\psi)}\left(1-e^{-2H(y,\psi)}\right)^{\theta-1}e^{-\beta\left(1-e^{-2H(y,\psi)}\right)^{\theta}}}{(1-e^{-\beta})}\right\} \quad (61)$$

4. SOME FAMILY MEMBERS OF TELTL-G

The TELTL-G family's unique sub-models, the Truncated exponential log topp-leone exponential Distribution (TELTL-ED) and the Truncated exponential log topp-leone weibull Distribution (TELTL-WD), are addressed in this section.

4.1. Truncated exponential log topp-leone exponentiated distribution.

Let $H(y,\psi)$ be the cdf of the exponential random variable given by $G(y,\psi) = 1 - e^{-\pi y}, y, \pi > 0$ and $g(y,\psi) = e^{-\pi y}$. Then, the cdf of TELTL-E distribution has the following form;

$$F_{TELTL-E}(y,\beta,\theta,\pi) = \frac{1-e^{-\beta\left(1-e^{-2(1-e^{-\pi y})}\right)^{\theta}}}{1-e^{-\beta}} \quad (62)$$

And the corresponding probability density function is;

$$f_{TELTL-E}(y,\beta,\theta,\pi) = \frac{2\beta\theta\ e^{-\pi y}\left(1-e^{-2(1-e^{-\pi y})}\right)^{\theta-1}}{1-e^{-\beta}}\text{x}$$
$$e^{-2(1-e^{-\pi y})}e^{-\beta\left(1-e^{-2(1-e^{-\pi y})}\right)^{\theta}} \quad y,\theta,\beta > 0 \quad (63)$$

While other properties of TELTL-E distribution including quantile, survival function, and Hazard rate function are;

$$y_\alpha = -\frac{1}{\pi}log\left\{1+\frac{1}{2}log\left\{1-\left\{-\frac{\log(1-\alpha(1-e^{-\beta})}{\beta}\right\}^{\frac{1}{\theta}}\right\}\right\} \quad (64)$$

$$S(y) = \frac{e^{-\beta\left(1-e^{-2(1-e^{-\pi y})}\right)^{\theta}} - e^{-\beta}}{1-e^{-\beta}} \quad (65)$$

$$H(y) = \frac{2\beta\theta\ e^{-\pi y}\left(1-e^{-2(1-e^{-\pi y})}\right)^{\theta-1}e^{-2(1-e^{-\pi y})}}{e^{-\beta\left(1-e^{-2(1-e^{-\pi y})}\right)^{\theta}} - e^{-\beta}}\text{x}$$
$$e^{-\beta\left(1-e^{-2(1-e^{-\pi y})}\right)^{\theta}} \quad (66)$$

The pdf plot of the TELTL-E distribution for different values of the parameters is illustrated in Figure (1/a), which shows that the distribution has a positive skewed distribution with a monotonic increasing cdf in Figure (2/b) for different values of the parameter, showing an increasing upward and constant at 1. While Figure (3/c) demonstrates an increase in the hazard function and a decrease as x tends to zero, and the survival function as well in Figure (4/d).

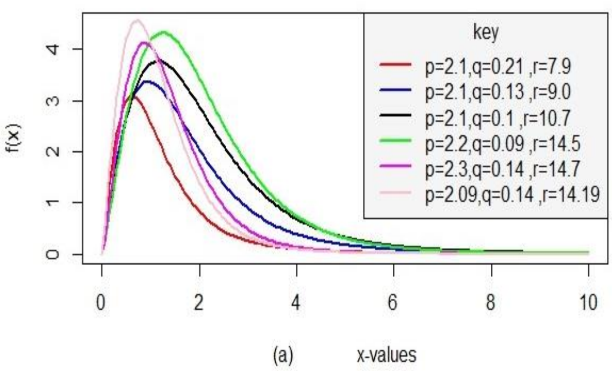

Figure 1: pdf plot for some values of parameters of TELTL-E

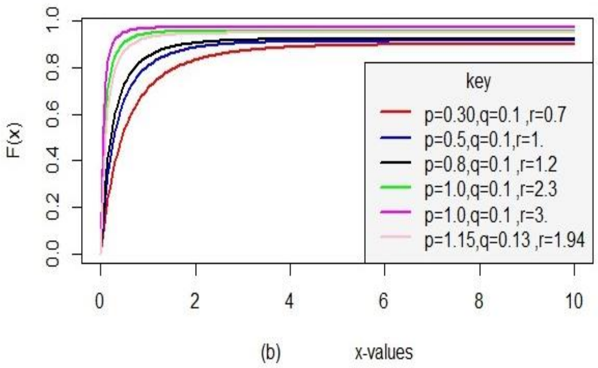

Figure 2: cdf plot for some values of parameters of TELTL-E

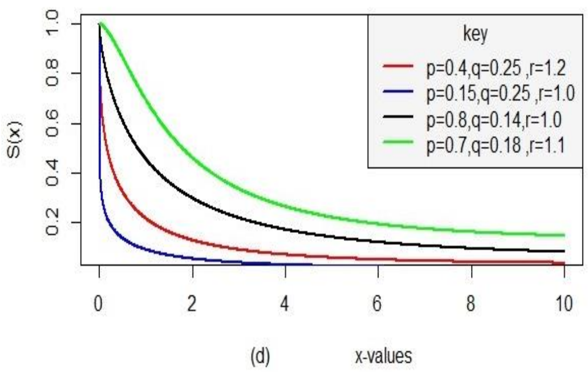

Figure 3: survival plot for some values of parameters of TELTL-E

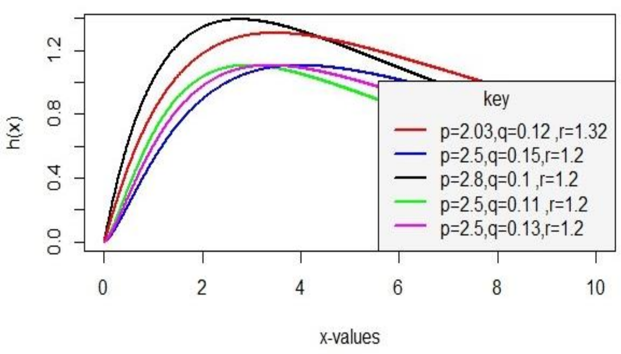

Figure 4: Hazard rate plot for some selected values of parameters of TELTL-E

4.2. Truncated exponential log topp-leone weibull distribution

Let $H(y,\psi)$ be the cdf of the weibull random variable given by $G(y,\psi) = 1 - e^{-\left(\frac{y}{\zeta}\right)^{\sigma}}, y,\zeta,\sigma > 0$ and $g(y,\psi) = e^{-\left(\frac{y}{\zeta}\right)^{\sigma}}$. Then, the cdf of TELTL-E distribution has the following form;

$$F_{TELTL-W}(y,\beta,\theta,\zeta,\sigma) = \frac{1-e^{-\beta\left(1-e^{-2\left(1-e^{-(y/\zeta)^{\sigma}}\right)}\right)^{\theta}}}{1-e^{-\beta}} \quad y,\beta,\theta,\zeta,\sigma > 0 \qquad (67)$$

$$f_{TELTLW}(y,\beta,\theta,\zeta,\sigma) = \frac{\left(1-e^{-2\left(1-e^{-(y/\zeta)^{\sigma}}\right)}\right)^{\theta-1}}{1-e^{-\beta}} \text{x}\; 2\beta\theta e^{-(y/\zeta)^{\sigma}} e^{-2\left(1-e^{-(y/\zeta)^{\sigma}}\right)} e^{-\beta\left(1-e^{-2\left(1-e^{-(y/\zeta)^{\sigma}}\right)}\right)^{\theta}} \quad y,\theta,\beta,\zeta,\sigma > 0 \qquad (68)$$

Likewise, we can obtain other properties including quantile function, survival function, and Hazard rate function as follows;

$$y_k = \zeta\left\{-log\left\{1+\frac{1}{2}log\left\{1-\left\{-\frac{\log\left(1-k\left(1-e^{-\beta}\right)\right)}{\beta}\right\}^{\frac{1}{\theta}}\right\}\right\}\right\}^{\frac{1}{\sigma}} \qquad (69)$$

$$S(y) = \frac{e^{-\beta\left(1-e^{-2\left(1-e^{-(y/\zeta)^{\sigma}}\right)}\right)^{\theta}} - e^{-\beta}}{1-e^{-\beta}} \qquad (70)$$

$$H(y) = \frac{e^{-(y/\zeta)^{\sigma}}\left(1-e^{-2\left(1-e^{-(y/\zeta)^{\sigma}}\right)}\right)^{\theta-1}}{e^{-\beta\left(1-e^{-2\left(1-e^{-(y/\zeta)^{\sigma}}\right)}\right)^{\theta}} - e^{-\beta}} \text{x}\; 2\beta\theta e^{-2\left(1-e^{-(y/\zeta)^{\sigma}}\right)} e^{-\beta\left(1-e^{-2\left(1-e^{-(y/\zeta)^{\sigma}}\right)}\right)^{\theta}} \qquad (71)$$

For several values of the parameters, the pdf plot of the TELTL-W distribution is illustrated in Figure

(5/a), which shows that the distribution has a positive skewed distribution with a monotonic increasing cdf in Figure (6/b), showing an increasing upward and constant at 1. While Figure (7/c) demonstrates an increase in the hazard function and a decrease as x tends to zero, and the survival function as well in Figure (8/d).

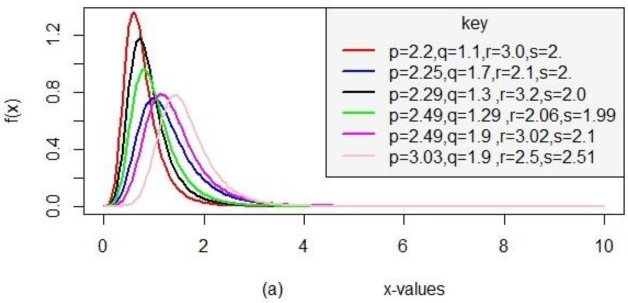

Figure 5: pdf plot for some values of parameters of TELTL-W

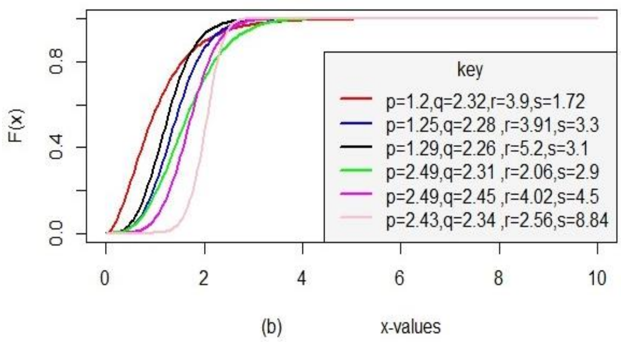

Figure 6: cdf plot for some values of parameters of TELTL-W

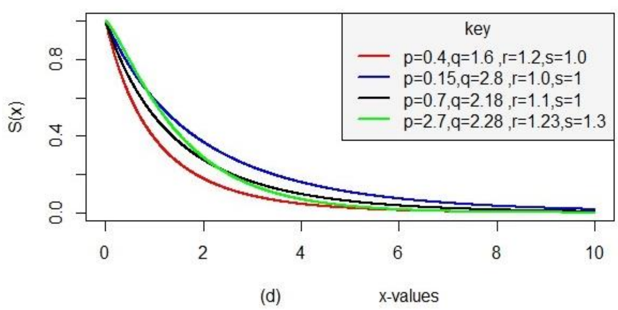

Figure 7: survival plot for some values of parameters of TELTL-W

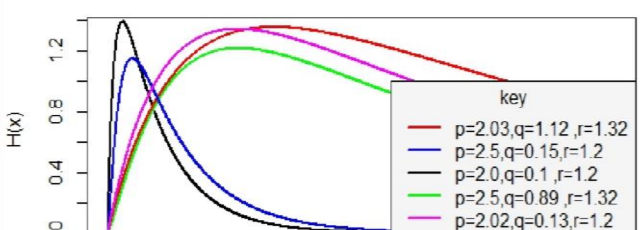

Figure 8: Hazard rate plot for some selected values of parameters of TELTL-W

5. APPLICATION

5.1. Competitors models

The following standard models used in comparing and evaluation the performance of the proposed sub-models of TELTL-G family of distribution, the models are;

- Zubair Weibull Distribution by [1]. The cumulative distribution is given by;

$$F(x) = \frac{e^{\alpha(1-e^{-\gamma x^{\theta}})^2}}{e^{\alpha}} \quad x, \alpha, \gamma, \theta > 0 \qquad (72)$$

- Kumaraswamy-Pareto Distribution proposed by [8] is;

$$F(x) = 1 - \left\{1 - \left[1 - \left[\frac{x}{u}\right]^r\right]^s\right\}^t \; x, r, s, t > 0 (73)$$

- Rayleigh Pareto Distribution by [3];

$$F(x, p, b, c = 2\theta) = 1 - \left(\frac{x}{b}\right)^c e^{-\frac{1}{2b^2}} \quad x, p, b, c > 0 \qquad (74)$$

- The odd generalized exponential weibull distribution by [23];

$$F(x) = (1 - e^{-\lambda(e^{\theta x^{\beta}} - 1)})^{\alpha} \; x, \lambda, \alpha, \theta, \beta > 0 (75)$$

- Exponentiated Weibull Distribution by [15]. The distribution is defined in the following way. It has cumulative distribution function given by;

$$Q(r) = (1 - e^{-(\theta r)^{\varepsilon}})^{k} \; r, \theta, \varepsilon > 0 \qquad (76)$$

5.2. Datasets for waiting time

The waiting periods (measured in seconds) between the 65 consecutive eruptions of the Kiama Blowhole are included in this data set. Jim Irish recorded the information on July 12, 1998, using a digital watch. Several publications have cited these data, including [16] and [5]. The real data are:

Table 1: Data sets for waiting time of successive eruptions of the Kiama Blowhole

83, 51, 87, 60, 28, 95, 8, 27, 15, 10, 18, 16, 29, 54, 91, 8, 17, 55, 10, 35, 47, 77, 36, 17, 21, 36, 18, 40, 10, 7, 34, 27, 28, 56, 8, 25, 68, 146, 89, 18, 73, 69, 9, 37, 10, 82, 29, 8, 60, 61, 61, 18, 169, 25, 8, 26, 11, 83, 11, 42, 17, 14, 9, 12.

Table 2: Summary of the waiting time datasets

Min	Q1	Median	Mean	Q3	Max
7.0	14.6	28	39.83	60	169

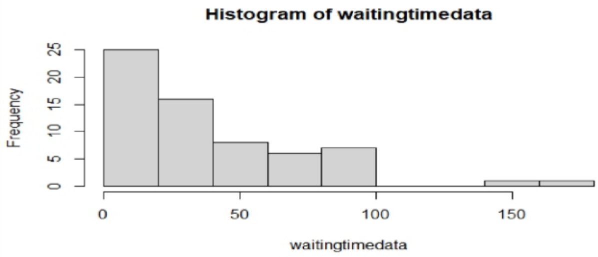

Figure 9: Histogram showing the skewness of waiting time data sets.

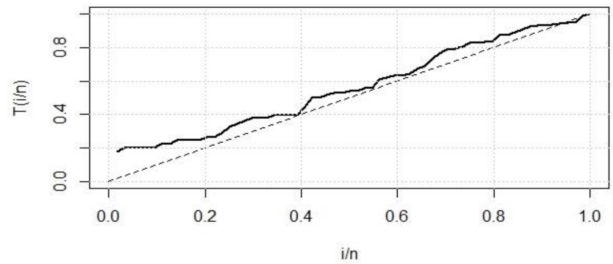

Figure 10: Histogram showing the TTT plots for model adequacy for waiting time data sets.

Calculations are made to compare the fitted models using the goodness-of-fit metrics, which include the log-likelihood function evaluated at the MLEs, the Akaike information criterion (AIC), the Bayesian information criterion (BIC), the Akaike information corrected criterion (AICc), Anderson Darling (A*), Cramer-von Mises (W*), and Kolmogorov-Smirnov (K-S*). The better the fit to the data, generally speaking, the smaller the values of these statistics [5].

Table 3: MLE's of the parameter(s)

Distribution	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$
TELTL-W	2.225	0.3829	0.00000021	0.1150
TELTL-E	1.88118	0.00628	0.36431	
OGEW	2.22827	0.43048	0.36757	0.1995
ExW	0.98523	0.3819	23.4871	
ZW	2.2263	0.7087	0.1708	
KwP	10.9202	2.6152	4.6173	0.3303
RP	0.00247	0.39282	0.05613	

Table 4: Goodness of fit measure for the fitted Models

Distribution	*LL*	AIC	AICc	BIC
TELTL-WEIB.	533.11	-1057.5	-1053.2	-1049.6
TELTL-EXP.	-25.48	56.955	57.36	63.43
OGEW	-103.78	215.55	216.23	224.19
EWD	-293.96	593.92	594.32	600.40
ZWD	-296.75	599.89	599.49	605.97
KwPD	-298.93	606.53	605.86	614.49
RPD	-379.80	765.59	765.99	772.07

Table 5: Normality test for the fitted models

Distribution	KS*	A*	W*
TELTL-WD	1	8.4728	1.4627
TELTL-ED	0.75866	12.095	2.4103
OGEWD	0.99126	1.4572	0.22517
EWD	0.094259	0.81842	0.11201
ZWD	0.20814	1.5251	0.23675
KwPD	0.10326	0.86123	0.11965
RPD	0.96038	0.79	0.10793

5.3. Data sets for Wheaton River flood

The information relates to the Wheaton River's flood peaks (measured in m3/s) in Yukon Territory, Canada, near Carcross. The data, which is rounded to one decimal place, includes 72 exceedances for the years 1958–1984. The data were used by [8].

Table 6: Data sets of Wheaton River flood

1.7, 2.2, 14.4, 1.1, 0.4, 20.6, 5.3, 0.7, 1.9, 13.0, 12.0, 9.3, 1.4, 18.7, 8.5, 25.5, 11.6, 14.1, 22.1, 1.1, 2.5, 14.4, 1.7, 37.6, 0.6, 2.2, 39.0, 0.3, 15.0, 11.0, 7.3, 22.9, 1.7, 0.1, 1.1, 0.6, 9.0, 1.7, 7.0, 20.1, 0.4, 2.8, 14.1, 9.9, 10.4, 10.7, 30.0, 3.6,5.6, 30.8, 13.3, 4.2, 25.5, 3.4, 11.9, 21.5, 27.6, 36.4, 2.7, 64.0,1.5, 2.5, 27.4, 1.0, 27.1, 20.2, 16.8, 5.3, 9.7, 27.5, 2.5, 27.0

Table 7: Summary of the Wheaton River flood data sets

Min.	Q1	Median	Mean	Q3	Max
0.10	2.13	9.50	12.204	20.3	64.0

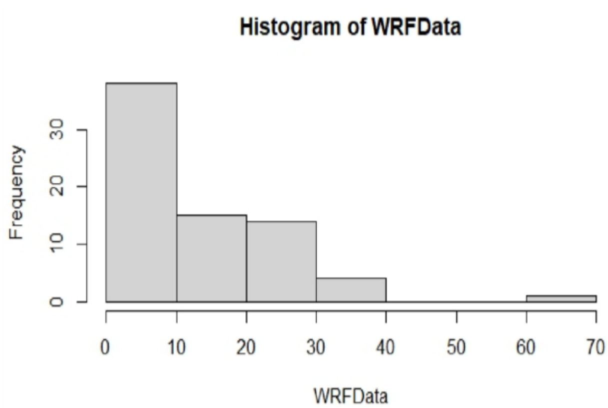

Figure 11: Histogram showing the skewness of

Wheaton River flood data sets.

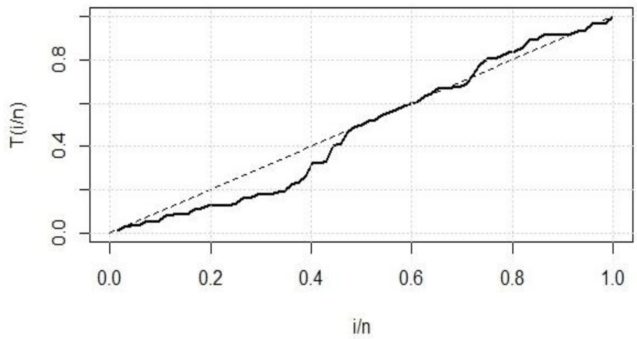

Figure 12: Histogram showing the TTT plots for model adequacy for Wheaton River flood data sets.

Like the previous data, Calculations are also made to compare the fitted models using the goodness of-fit metrics, which include the log-likelihood function evaluated at the MLEs, the AIC, BIC, AICc, A*, W*, and K-S*.

Table 8: MLE's of the parameter(s)

Distribution	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$
TELTL-WEIB	2.225	0.3794	0.000004046	0.2212
TELTL-EXP	0.99995	0.0000003	0.54444807	
EWD	0.05038	1.3832	0.5207	
ZWD	1.7716	0.5148	0.6364	
RPD	0.00247	0.4853	0.1236	

Table 9: Goodness of fit measure for the fitted models

Distribution	***LL***	**AIC**	**AICc**	**BIC**
TELTL-WEIB.	695.646	-1382.69	-1383.3	-1374.2
TELTL-EXP.	63.665	-130.977	-131,33	-124.50
EWD	-251.03	508.050	508.403	514.88
ZWD	-252.88	512.109	511.756	518.59
RPD	-284.73	575.468	575.82	582.30

Table 10: Normality test for the fitted models

Distribution	**KS***	**A***	**W***
TELTL-ED	0.57519	9.1135	1.7285
TELTL-WD	1	2.968	0.51964
EWD	0.10742	0.6358	0.10444
ZWD	0.28944	1.5209	0.26993
RPD	0.88035	1.4535	0.26069

5.4. Discussion

The analysis conducted demonstrate the comparison of the proposed models and other base line distribution using the skewed waiting time datasets and Wheaton River flood data sets as shown in Figure 1& 2 and the data summary in Table 2 & 7.Table 3 & 8 provides the parameter estimates for each of the data set's fitted distributions. Conversely, Table 4 & 9 listed each model's matching AIC, AICc, and BIC values. The table provides sufficient evidence to demonstrate that the sub-models of truncated Exponential log topp leone family of distribution (TELTL-G) outperforms the baseline distribution and certain of its extensions. Nonetheless, the fact that relatively little information was lost is explained by the negative values of the AIC, AICc, and BIC, as in TELTL-WD. As a result, when compared to the other five distributions used in the performance comparisons for fitting the same data set, it might be chosen as the best model as well as the TELTL-ED.

In order to determine the values of the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), and Consistent Akaike Information Criterion (AICc), log-likelihood was utilized to estimate the parameters of each distribution for the dataset. Tables 4 & 6 display the collected results. The sub-models of TELTL-GD yields the lowest values for the AIC, BIC, and AICc, as can be seen. Lastly, Table 5 & 10 display the values of Anderson-Darling (A*), Cramer-von Mises (W*), and Kolmogorov-Smirnov (K-S*). Thus, among the studied distributions, we conclude that the the sub-models of TELTL-GD offers the best fit compare to other models.

6. CONCLUSION

A probability model known as the Truncated Exponential Generalized Family of Distribution (TELTL-G) was created with two parameters and a parameter vector. The specific expressions for the suggested distribution’s moments, moment-generating function, quantile function, median, survival function, hazard function, and ordered statistics have all been carefully examined. Additionally, the distribution is favorably skewed, according to several graphs of the distributions. The estimation using maximum likelihood, goodness of fit (Cramer-Von-Mises), and least square method was used to estimate the parameters and parameter vector.

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Ethical Consideration

No any Ethical clearance required before the commencement of these research.

Contribution to Knowledge

It has been recommended that a novel generalized family of distributions is superior to various J-shape probability distributions already in use for modeling right-skewed data sets.

Areas for Further Research

Researchers with an interest in this field of study can examine the estimate of confidence intervals for the suggested distribution parameters. Additionally, for the sake of theoretical comparison and methodology validation, researchers might estimate the parameters of the new distribution through the use of regression and Bayesian methodologies.

Data availability statement

All data used in this work can be found within the manuscript.

References

[1] Ahmad, Z. (2020). The zubair-g family of distributions: properties and applications. Annals of Data Science, 7(2):195–208.

[2] Al-Babtain, A. A., Elbatal, I., Chesneau, C., and Elgarhy, M. (2020). Sine topp-leone-g family of distributions: Theory and applications. Open Physics, 18(1):574–593.

[3] Al-Kadim, K. A. and Mohammed, B. D. (2018). Rayleigh pareto distribution. Journal of University of Babylon for Pure and Applied Sciences, 26(1):84–95.

[4] Almetwally, E. M., Alharbi, R., Alnagar, D., and Hafez, E. H. (2021). A new inverted toppleone distribution: applications to the covid-19 mortality rate in two different countries. Axioms, 10(1):25.

[5] Aryal, G. R., Ortega, E. M., Hamedani, G., and Yousof, H. M. (2017). The topp-leone generated weibull distribution: regression model, characterizations and applications. International Journal of Statistics and Probability, 6(1):126–141.

[6] Bantan, R. A., Jamal, F., Chesneau, C., and Elgarhy, M. (2019). A new power topp–leone generated family of distributions with applications. Entropy, 21(12):1177.

[7] Boos, D. D. (1982). Minimum anderson-darling estimation. Communications in StatisticsTheory and Methods, 11(24):2747–2774.

[8] Bourguignon, M., Silva, R. B., Zea, L. M., and Cordeiro, G. M. (2013). The kumaraswamy pareto distribution. Journal of Statistical Theory and Applications, 12(2):129–144.

[9] Hassan, A. S., Elgarhy, M., and Ragab, R. (2020). Statistical properties and estimation of inverted topp-leone distribution. J. Stat. Appl. Probab, 9(2):319–331.

[10] Lawless, J. F. (2011). Statistical models and methods for lifetime data. John Wiley & Sons.

[11] Mahdavi, A. (2017). Generalized topp-leone family of distributions. Journal of Biostatistics and Epidemiology, 3(2):65–75.

[12] Merovci, F., Yousof, H. M., and Hamedani, G. (2020). The poisson topp leone generator of distri- butions for lifetime data: theory, characterizations and applications. Pakistan Journal of Statistics and Operation Research.

[13] Muhammad, M., Liu, L., Abba, B., Muhammad, I., Bouchane, M., Zhang, H., and Musa, S. (2023). A new extension of the topp–leone-family of models with applications to real data. Annals of Data Science, 10(1):225–250.

[14] Nadarajah, S. and Kotz, S. (2003). Moments of some j-shaped distributions. Journal of Applied Statistics, 30(3):311–317. 19

[15] Pal, M., Ali, M. M., and Woo, J. (2006). Exponentiated weibull distribution. Statistica, 66(2):139– 147.

[16] Pinho, L. G. B., Cordeiro, G. M., and Nobre, J. S. (2015). The harris extended exponential distri- bution. Communications in Statistics-Theory and methods, 44(16):3486–3502.

[17] Reyad, H. M., Alizadeh, M., Jamal, F., Othman, S., and Hamedani, G. G. (2019). The exponentiated generalized topp leone g family of distributions: properties and applications. Pakistan Journal of Statistics and Operation Research, pages 1–24.

[18] Saboor, A., Kamal, M., and Ahmad, M. (2015). A new lifetime distribution with bathtub hazard rate function. “All papers published in the, page 59.

[19] Sule, I., Sani, I. D., Audu, I., and Jibril, H. M. (2020). On the topp leone exponentiated-g family of distributions: properties and applications.

[20] Suleiman, A. A., Daud, H., Ishaq, A. I., Othman, M., Sokkalingam, R., Usman, A., and Osi, A. A. (2023). The odd beta prime inverted kumaraswamy distribution with application to covid-19 mortality rate in italy. Engineering Proceedings, 56(1):218.

[21] Swain, J. J., Venkatraman, S., and Wilson, J. R. (1988). Least-squares estimation of distribution functions in johnson's translation system. Journal of Statistical Computation and Simula- tion, 29(4):271–297.

[22] Tahir, M. H. and Cordeiro, G. M. (2016). Compounding of distributions: a survey and new gener- alized classes. Journal of Statistical Distributions and Applications, 3:1–35.

[23] Tahir, M. H., Cordeiro, G. M., Alizadeh, M., Mansoor, M., Zubair, M., and Hamedani, G. G. (2015). The odd generalized exponential family of distributions with applications. Journal of Statistical Distributions and Applications, 2:1–28.

[24] Topp, C. W. and Leone, F. C. (1955). A family of j-shaped frequency functions. Journal of the American Statistical Association, 50(269):209–219.

[25] Yahaya, A. and Abba, B. (2017). Odd generalized exponential inverse-exponential distribution with its properties and application. Journal of the Nigerian Association of Mathematical Physics, 41:297–304.

[26] Yousof, H. M., Alizadeh, M., Jahanshahi, S., Ramires, T. G., Ghosh, I., and Hamedani, G. (2017). The transmuted topp-leone g family of distributions: theory, characterizations and applications. Journal of Data Science, 15(4):723–740. 11 20